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VISUALIZATION OF ALGEBRARY IDENTIFICATIONS IN RXR=R^2 (PLANE) AT SECONDARY LEVEL*

Birol TEKİN
Assist. Prof. Dr., Amasya University, Amasya, Turkey, biroltekin@amasya.edu.tr
ORCID: 0000-0001-6828-5274

Alper Cihan KONYALIOĞLU
Prof. Dr., Kazım Karabekir University, Erzurum, Turkey,ackonyali@atauni.edu.tr
ORCID: 0000-0002-6009-4251

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ABSTRACT

It is known that mathematics has an important effect on the development of science and technology. Identities, which are widely used in many branches of science, form the basis of some mathematics courses such as algebra. However, it appears as one of the learning areas of secondary school 7th grade mathematics and is seen as one of the difficult subjects to learn. The fact that there are many and abstract formulas about identities is difficult for students to remember. Therefore, visualization is important in teaching abstract concepts in a meaningful way and in associating concepts with daily life. Visualization is the use of mathematical and geometric symbols, shapes, diagrams, etc. to make something unseen imaginable. It is the drawing or showing of the shape and picture of an abstract concept with the help of pencil and computer. Visualization and visual proof are central to mathematics. The visualization approach is used in mathematics and mathematics education as a tool, not as a goal in the lessons. Identities are seen as one of the learning areas that we encounter in the solution of problems that we encounter in our daily lives and that are difficult to learn. In this direction, the visual proofs of the identities in R2 are shown in detail to the researchers thanks to the visualization approach, so that the algebraic identities can be learned more easily. Although identities are an important learning area in mathematics and algebra lessons, they are thought to be difficult to understand and learn by students. In line with this idea, students have difficulty in understanding identities as the visual models from which identities are obtained are not adequately expressed by the teachers and are not associated with daily life. For this reason, it is suggested by mathematics teachers that as a solution to this problem, visualizing abstract mathematical concepts in lessons or showing algebraic and visual proofs of some theorems may be beneficial. In this study, which was planned as a theoretical study, Within the scope of descriptive scanning model, document analysis technique was used. Visual proofs of basic algebraic identities are shown to researchers in detail by using geometric figures. In line with this purpose, some suggestions have been made to shed light on this issue for interested researchers.

Keywords: Algebraic identities, visual proof, visualization

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INTRODUCTION

In the 21st century, depending on the developments in science and technology, it has led to changes in the aims of mathematics education. In line with this purpose, the importance of mathematics and its use in daily life have gained importance. Mathematics has an important effect on the development of science and technology. Mathematics forms the basis of all sciences. The rapidly developing science and technology suggests the emergence of new teaching methods and techniques in the field of education. In this context, the visualization approach plays an important role as a new teaching method. It is very important for students to make algebraic or visual proofs of some mathematical concepts and theorems in lessons, because what is learned visually provides an easier and permanent learning. Learning based on cause-effect relationships and logical answers to these questions ensure that information is learned by making sense of it. One of the learning areas where such relationships are most common is "proofs". While proving and proving skills are seen an important topic in mathematics and mathematics education, it is considered as one of the most difficult mathematical skills to learn and teach. Basically, the method of proof, which is based on the idea of "seeing is believing", has a great importance for mathematicians. When the literature is examined, different types of proof are encountered. We can define it as the demonstration of the truth or falsity of a proposition by scientific methods

According to Hersch (1993), the word proof means showing, trying and testing the correctness of a situation or event. The main aim of a mathematical proof is to show why a proposition is true or false (Hanna, 2000). Mathematical thinking skills of students are improved by proofs made in mathematics and geometry lessons. For this reason, proof is at the center of mathematics and mathematics education and is used in lessons at every stage of education (Polat and Demircioğlu, 2021). Proof is considered an important tool that enables us to learn mathematics in a meaningful way in the context of conceptual knowledge (Knuth, 2002). Proof has a significant impact in students' ability to construct their mathematical knowledge in the context of conceptual knowledge in their minds. Therefore, proof is included in the mathematics curriculum of many countries (Cited by Winner and Battista, 2022). They stated that proof, which is considered as one of the main subjects of performing mathematics, providing communication and recording, is nothing but doing mathematics (Joseph and Schoenfeld, 2020). Dogan and William-Pierce (2021), in their study, stated that wordless proofs act as a bridge in visual and deductive proofs of a theorem, which is considered as one of the important research areas. They stated that geometric proofs are important in terms of providing any proof, understanding, reasoning and development of students' skills and it would be beneficial to conduct more research on this subject (Battista, 2007; Cirillo & May, 2020; Sinclair et al., 2017). In the study of Mirza, et al. (2022), all participants stated that proof "takes the task of proving or constructing a proposition as a tool". The common feature of all sciences is to prove the the correctness of the data determined by experiments and observations by scientific methods. In this context, a mathematician demonstrates the correctness of mathematical expressions and propositions by making use of logical inferences (Hanna & Barbeau, 2002; Polster, 2004).

In order to learn mathematics better in the context of conceptual knowledge, Since proof is at the center of mathematics, it is necessary to demonstrate proofs in their classes. In fact, many students do not know why they need to prove in mathematics lessons (Tekin & Konyalıoğlu, 2010). Dede (2013) defined it as "the content that a mathematical knowledge shows is the one that is defined as valid by mathematicians" and stated that this is accepted as a mathematical proof. When we examine the curriculum of mathematics lessons, many theorems such as identities are proved. Identities, which facilitate the solution of many problems, are one of the learning areas of the secondary school mathematics curriculum. When students study basic algebraic identities, they often memorize and apply these formulas to solve routine exercises. Sometimes students have difficulty remembering identities. For this reason, algebraic proofs or especially "visual proofs" can be useful for students to understand basic identities better. Visual proofs or presentations are of considerable benefit in concretizing intangible mathematical concepts. In mathematics and mathematics education, the concretization of concepts and the use of visualization approach have an important place. The use of visualization approach in teaching identities increases the quality of teaching and allows students to realize some details in the process of teaching concepts by using visual models (Özdemir, Duru, & Akgün, 2005). It is important for mathematical literacy to show the correctness of algebraic expressions to explain the rules of logic, induction and deduction concepts by using the visualization approach (Özer & Şan, 2013).

Visual representations such as visual models and pictures used to understand fractions are of considerable benefit in visualization and easier understanding of intangible mathematical concepts that students encounter (Shin & Bryant, 2015). The importance of studies in this field has increased in the last 20 years because computers have had a significant impact on visualization. In visualization, it is important whether or to what extent there are visual representations. 90 % of the input into our brain is visual, and approximately 65 % of the people in the society are those who are predisposed to visual learning (McCoy, 2019).

Visualization

Representations and visualizations make it easier to understand and remember abstract concepts in mathematics, which is seen as a course of concentrated intangible concepts. In order for invisible, mathematical concepts to be understood more easily, intangible concepts must be imaginable and transferable to a concrete plane (Arcavi 2003).Concretization is a teaching principle used frequently so that the information can be perceived by the five senses. Visualization makes both an object and an action visible. In other words, visualization is a method that enables the transfer of something from invisible to a visible in an easily understandable state (Arcavi,2003; Bagni 1998; McCormick et al.,1987). A more detailed definition of visualization “To develop previously unknown ideas and to improve thinking and understanding about these ideas; It is the process and ability of making information more understandable by interpreting and reflecting through visual designs such as pictures, images, diagrams, etc.” (Cited by Tekin, 2010).In her study, Doyuran (2014) explained that students' problems in using geometric shapes and associating mathematical symbols with each other while solving problems related to daily life can be solved by means of visualization approach. Although there are studies related to the visualization of different concepts, there are also studies related to the affective and cognitive characteristics of students. The lessons they took and the use of visualization approach in the lessons determined that there was a positive result in the development of students' ability to think abstractly (Cited by İlhan and Tutak,2021). Problem-Based Learning Approaches (PBLA) related to visualization and visualization technique were used to determine the improvement in students' Attitudes Towards Learning Mathematics and its components. They found that the visualization approach was beneficial because there was a significant relationship between students' attitudes (Abate, Atnafu & Michael,2022). They expressed the phenomenon called visualization as the ability, process and product of creating, interpreting, using and reflecting pictures, images, diagrams using technological tools or pencils on paper in our minds in order to describe, think and develop a concept or information (Uysal-Koğ & Başer, 2012).

Mathematics lessons supported with visualization tools enable students to improve their computation and remembering skills. It also helps students who are not good at visualizing, modeling, and thinking about abstract concepts to understand better. Polya is one of the important scientists who established a bridge between visualization and problem solving in mathematics teaching. Polya, by saying “draw a figure”, he emphasizes that when solving a mathematical problem, it may be an important start to draw a shape suitable for the structure of the problem. Because, firstly, the shape to be drawn according to the problem makes it easier to understand the problem and find the solution. In her study investigating the effect of visualization on visualization and problem solving skills, it is emphasized that visual tools have been used by mathematicians for centuries and a mathematician who is good at problem solving can use visual tools effectively (Stylianou, 2002). Visualization is used for different aim in mathematics education. Dufour-Janvier et. al. (1987) expressed them as follows (Özgün-Koca,1998):

- Visual representations are at the core of mathematics.
- Visual representations allow abstract mathematical concepts to be embodied.
- Visual representations are used to facilitate the teaching of some difficult mathematics subjects.

Visual representations have a significant impact in the students' positive attitude towards the lesson and the easier understanding of the lesson. Eroğlu, Moralı and Uğurel (2006), Konyalıoğlu (2003) also found that visualization positively affects student achievement in their studies. In the studies of some researchers, They stated that the use of visualization approach in lessons significantly increases academic achievement compared to other methods (Aktaş, 2006; Budanur, 2004; Demirel, 1996; Eritici, 2005). As Boaler et al. (2016) revealed in their study, studies on the brain showing the importance of visual thinking also supported the changes in the learning styles of students in schools.

Significance and Purpose of the Study

Identities are thought to be one of the difficult subjects to learn in mathematics, which is widely used in some courses taught in mathematics, physics and engineering departments. After students face with their identities for the first time while studying in the 7th grade of secondary school, they repeat the same subject in high school and university education. At the same time, It provides convenience in solving some of the problems we face in our daily lives. In particular, it has important areas of use in many subjects of mathematics and in many branches of mathematics related to mathematics. Identities have an important aim in creating the infrastructure of algebra courses. In this study, it is thought that the visualization of algebraic identities at secondary education level will be useful for students to learn intangible algebra concepts more easily. Therefore, it is aimed to show the algebraic proofs of algebraic identities and their proofs with visual figures in an understandable way and to introduce them to other researchers.

METHOD

This study, which uses document analysis review, includes the analysis of published documents containing information about concepts or phenomena intended for document review research. In the document analysis method, documents and facts are researched in detail and information about them is collected, or by making use of written materials given or presented as written and visual resources, it provides access to information showing the proof of something (Yıldırım and Şimşek, 2021).

Visual Proofs of Algebraic Identities

For a real number value not equal to zero, both sides of the equation are always true, called "identity". Algebraic identities are widely used in many fields of mathematics, geometry, physics, and engineering and help to solve many problems we encounter in our daily lives. Identities are introduced to students for the first time in secondary school and then they continue to study at high school and university and widely used in solving many problems that students can face in their daily lives. The visualization approach has an important effect on proving abstract mathematical concepts and theorems. Visual proofs do not contain detailed verbal explanations, only visual drawings, symbols or models. This makes the proof easy to understand. Borwein and Jögenson (1997) changed the differences between algebraic proof and visual proof in their study. In algebraic proof, there are a series of steps and logical inferences that point to the correctness of these steps. The visual proof must have the properties of the algebraic proof. The more visual models and concrete examples are used in teaching mathematical concepts, the easier it is for students to learn (Barth & Demirtaş, 1997). In another study conducted by Barth and Demirtaş (1997), the realization of teaching through concrete examples and visual materials enables students to learn easily by making sense of the information and to make the information permanent. Visual proof is the demonstration of the correctness of a proposition or theorem using figures and diagrams without using words (Nelsen, 2000).

Borwein and Jögenson (1997) state in their study that a good visual proof must satisfy three conditions.
These are;

- **Reliability:** the proof of a theorem must be acceptable to everyone. When proofs are made by different people, contradictory results should not occur.
- **Consistency:** For a proposition to be considered valid and true, it must be consistent first. When the proof of any theorem is done by different people, the truth-values of the proofs must be the same.
- **Repeatability:** The proof should be performed by different people and its accuracy should be shown.

In the visual proofing process, it is useful to consider the above-mentioned features. Yalın (2001) emphasizes that well-designed teaching tools and materials have positive effects on teaching processes and enrich learning processes. In addition, according to statistics obtained from another study, 65 % of the population consists of visual learners (Zopf et al., 2004). In a study by Kim and et al (2006), It was concluded that the number of visual learners in university students is 80% higher than verbal learners. While the debates about the importance of visual proofs continue, researches on their effectiveness are still on going. They emphasized in their studies that it is important for teachers to use visual shapes or models in lessons so that students can better understand the lessons (Hanna & Sidoli, 2007). Gierdien (2007) emphasizes that the visualization in its final form can be considered as a product while making non-verbal proofs. Visual representations have an important effect on the concretization of abstract mathematical concepts. Visual presentations or visual proofs supply to understand abstract concepts more easily in learning in terms of ensuring the retention of knowledge in learning using the sense organs (Tekin & Konyalıoğlu, 2010). Algebraic identities are considered a key element in forming the basis for mathematics and other courses from primary education to university (Muchoko, Jupri & Prabawanto, 2019). For this reason, teaching identities more meaningfully by visualizing is important in terms of ensuring the permanence of knowledge. In terms of learning, “A shape is more effective and meaningful than a thousand words” (Martin, 2020). In this case, visualization and visual proofs have a significant impact in teaching abstract concepts and making proofs of theorems.

Algebraic identities are used to facilitate the solution of some problems in primary school mathematics lessons. Therefore, it is useful to benefit from visual shapes in the teaching of lessons (Birgin & Demirören, 2020).Visual proofs of algebraic identities can be proved by using visual materials. The lgebraic and visual proofs of each one of the basic algebraic identities are presented below respectively step by step.

Visual proof of $(m +n)^2 = m^2 + 2.m.n + n^2$ perfect square identity

Step 1

Let’s take into consideration the square ABCD in figure 1, whose side lengt

Step 1

Let’s take into consideration the square ABCD in figure 1, whose side length is (m +n) unit. It is drawn a square with the side lengths of m and n unit starting from the corners B and D. In this case, one identical rectangle was formed at vertices A and C. The area of square ABCD was calculated as Area (ABCD) = |AB|.|AD| = (m + n). (m +n) unit square.

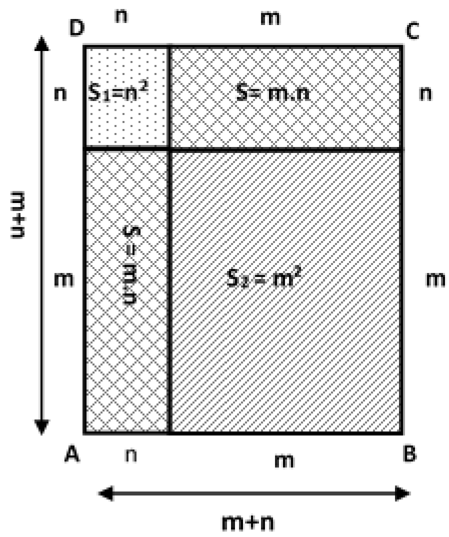

Figure 1. The Visual Represantation of the Identity $(m+n)^2 = m^2+2.m.n+n^2$

Step 2

Considering two identical rectangles with two different squares showing figure 1 and arranging them as follows, the total area is calculated by finding the area of each geometric figure, respectively. After that, figure 1 consists of four parts, two square regions and two identical rectangular regions (figure 2). The sum of the areas of the square and rectangular regions in figure 2 was calculated as $S_T = S_2+S+S+S_1 = m^2 + 2.m.n + n^2$

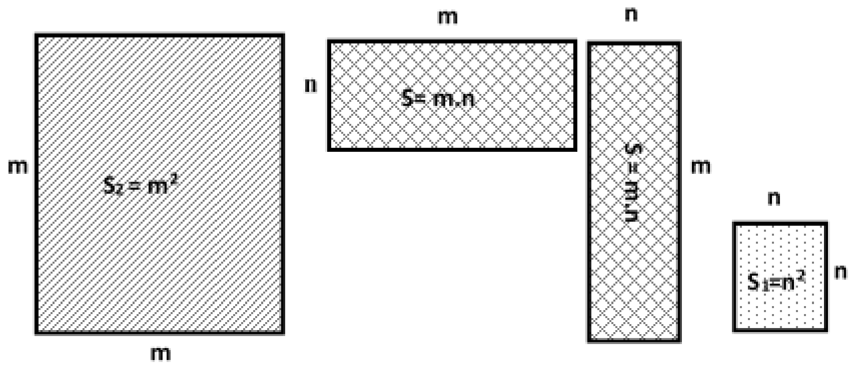

Figure 2. The Visual Represantation of the Areas of Two Identicial Rectangles and Two Different Squares

Step 3

The area of the square ABCD given in figure 1 (Step 1), whose side length is (m+n) unit, is equal to the sum of the areas of the squares and rectangles given in figure 2.

On the other hand, this area calculated in figure 2 (Step 2) is equal to the area of the square given in figure 1 (Step 1) side length (m +n) unit at the beginning. This explanation could be written algebraically as

Area (ABCD) = (m +n).(m +n) = $(m+n)^2 = S_T = S_2+S+S+S_1 = m^2 + 2.\,m.n + n^2$

Therefore, the perfect square identity $(m + n)^2 = m^2 + 2.m.\, n + n^2$ was obtained by visually.

Algebraic proof of the perfect square identity $(m + n)^2 = m^2+2.m.n + n^2$

We can also write as $(m + n)^2 = (m +n).\,(m +n)$, then, by applying the distributive property of multiplication over addition;

$(m + n).\,(m +n) = m.\,(m +n) + n.\,(m + n) = m^2+m.n+n.n+n^2 = m^2+2.m.n+n^2$

As a result; $(m + n)^2 = m^2+2.m.n+n^2$ was shown algebraically.

Visual proof of $(m-n)^2 = m^2 - 2.m.n + n^2$ perfect square identity

Step 1

Let's draw a side length m unit square in figure 3.

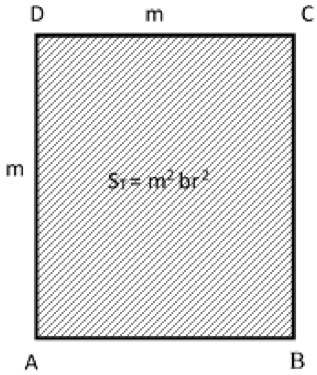

Figure 3. The Visual Represantation of the Area of Square with Sides m unit

Step 2

When a parallel line segment is drawn n unit away from the sides DC and AD, two identical rectangles and two squares with different side lengths are formed. The side lengths and areas of the geometric shapes formed in the ABCD square are written respectively (Figure 4). At the D and B corners, respectively; let's create two squares with side lengths n and (m-n) unit and two identical rectangles with side lengths n and (m-n) unit. The side lengths and areas of the geometric shapes formed in the ABCD square are written respectively.

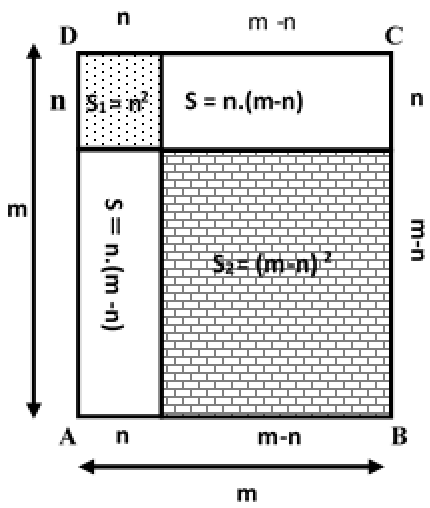

Figure 4. The visual Represantation of the Area of Two Different Squares and Two İdenticial Rectangles

Step 3

From the area of square ABCD given in figure 3, subtract the area of two identical rectangles and the area of a square with side length n unit. We can see this situation visually, in detail, in figure 5.

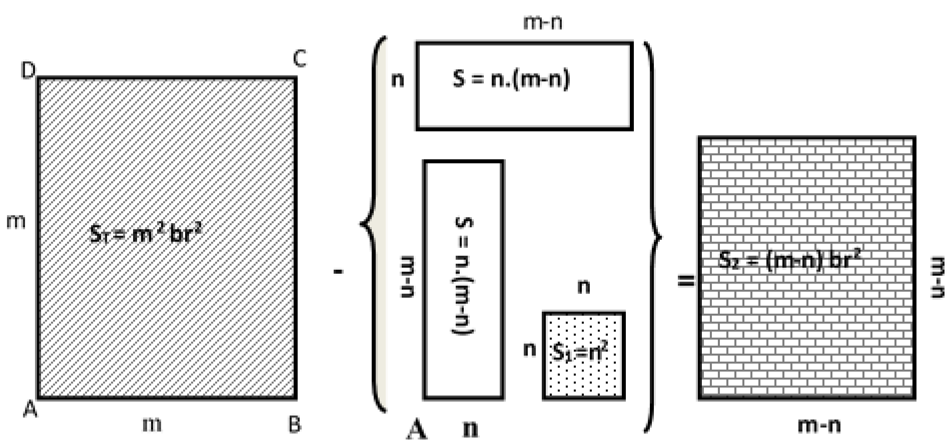

Figure 5. The Visual Represantation of the Area of Obtaining the Square with the Sıde (m-n) unit from Figure 4

This explanation (given by step 3) can be written algebraically as

Area (ABCD) – (2.S +S_1) = S_2 =m^2 – [2. n. (m-n) + n^2] = (m-n). (m -n) = $(m-n)^2$

So, the identiy $(m-n)^2 = m^2 - 2.m.n + n^2$ is obtained

Algebraic proof of the perfect square identity $(m-n)^2 = m^2-2.m.n+n^2$

This formula was shown algebraically as $(m-n)^2$ = (m-n). (m-n) by applying the distributive property of multiplication over subtraction.

$(m - n).(m - n) = m.(m-n) - n.(m-n) = m^2 - m.n - n.m + n^2$

As a result; $(m - n)^2 = m^2 - 2.m.n + n^2$ was shown algebraically.

$m^2 - n^2 = (m + n).(m-n)$ visual proof of two square difference identity

Step 1:

Let's draw firstly a square with a side length of m unit in figure 6.The area of square ABCD is m^2 br^2

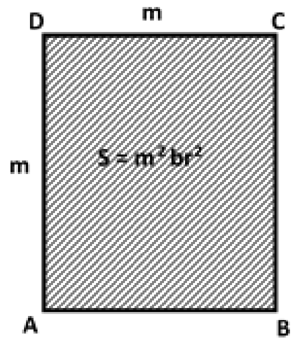

Figure 6. The area of Square with Side m unit

Step 2

After that, let's draw secondly a square EBGF with n unit from B corner of a square ABCD with a side length of m unit given in figure 7.From the B corner of ABCD square, a square with side length n unit was drawn (figure 7).

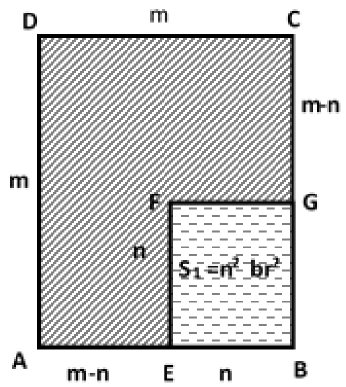

Figure 7. The square with Side n unit is obtained by at the Corner B square with Side m unit

Step 3

The square drawn in figure 7 is cut away or discarded using scissors as in figure 8. The area of the shape remaining after discarding the square with side length n unit.

This explanation could be written as Area (ABCD) – Area (EBGF) = (m^2- n^2) br^2

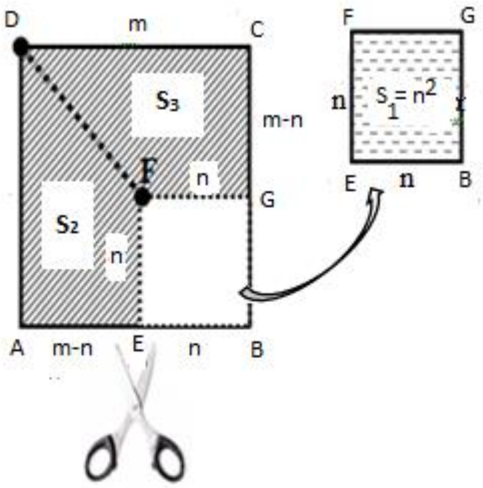

Figure 8. The Visual Area of the Shape Remaining after Discarding the Square with Side Length n unit.

Step 4

In figure 8, after the square taken, two perpendicular trapezoids with identical areas remain as seen figure 9.

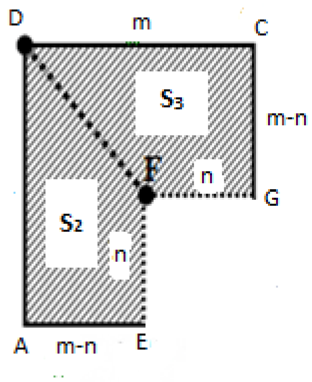

Figure 9.The Visual Represantation of the Remaining Picture Which is Called with the Same Area of Identicial Right Trapezoids

Step 5

A rectangle is formed by combining identical right two trapezoids given in figure 9, as in figure 10. Area of the generated AGCE rectangle was calculated algebraically as

Area (AGCE) = | AE |. | AG |= (m-n).(m +n) br^2

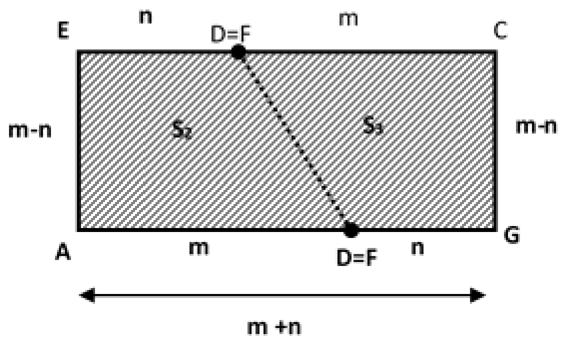

Figure 10. The Visual Represantation of Identicial Right Two Trapezoids Which is Combined to Obtain a Rectangle with Sıdes m+n and m-n

Step 6

The area in figure 8 (Step 3) is equal to the area of the AGCE rectangle given in figure 10 (Step 5). This explanation could be written algebraically as

Area (ABCD) – Area (EBGF) = Area (AGCE) = |AE|. | AG |

The values found are written in the algebraic formula given above. Then,

The proof of $(m^2 - n^2) = (m-n).(m+n)$ was completed visually.

The algebraic proof of identity of two squared difference: $m^2 - n^2 = (m+n).(m-n)$

If we multiply the expression (m +n). (m-n) by applying the distributive property of multiplication over subtraction;

$(m + n).(m - n) = m.(m-n) + n.(m-n) = m^2 - m.n + m.n - n^2 = m^2 - n^2$ was obtained.

As a result; $(m + n).(m - n) = m^2 - n^2$ was shown algebraically.

Visual proof of the identity $(m + n)^2 - (m-n)^2 = 4.m.n$

Step 1

It is drawn four identical rectangles with side lengths m and n unit at the corners, or place them as a model, so that they are in the inner region of the square ABCD, whose side length is (m +n) unit as seen figure 11.

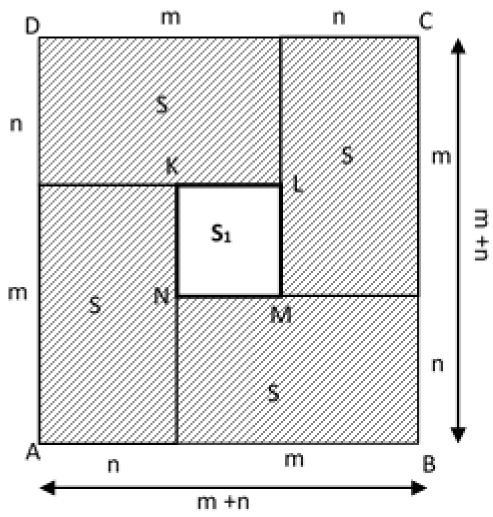

Figure 11. The Visual Represantation of Identicial Four Rectangles Which are Put with a Model or Drawn in the ABCD Square.

Step 2

When drawing or placing as a model as in figure 11, a square with side length (m-n) unit will be formed in the middle of square ABCD as seen in figure 12.

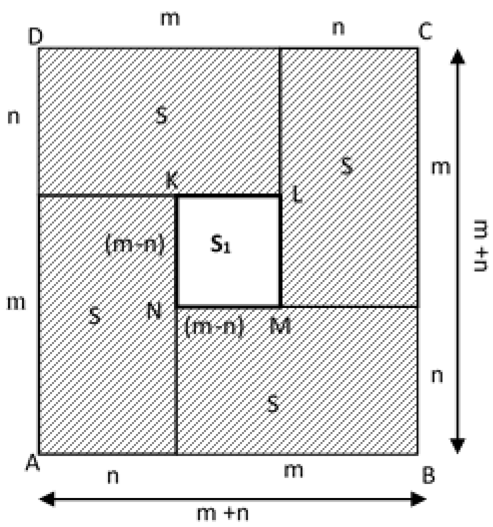

Figure 12. The Visual Represantation of the (m-n) Side Square Which is Obtained in the Middle of ABCD Square

Step 3

If we subtract the area of the square KLMN from the area of square ABCD, the remaining area is equal to the area of four identical rectangles as seen figure 13. If the values in the formula Area (ABCD) – Area (KLMN) = 4.S are replaced respectively in the formula. The proof of $(m + n)^2 – (m- n)^2 = 4.m.n$ was shown visually in figure 13.

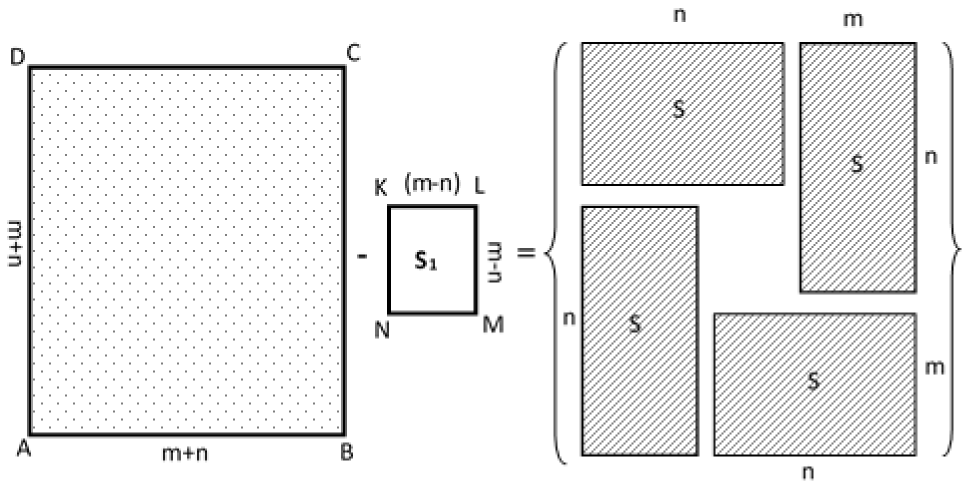

Figure 13. The Visual Proof of $(m+n)^2 – (m-n)^2 = 4.m.n$

Algebraic proof of the identity $(m+n)^2 – (m-n)^2 = 4.m.n$

This $(m+n)^2 – (m-n)^2 = 4.m.n$ algebraic formula could be written as

$(m + n). (m + n) – (m-n). (m -n) = 4.m.n$, by applying the distributive property of multiplication over addition and subtraction;

$(m + n). (m + n) – (m-n). (m –n) = m^2+2m.n+n^2-m^2+2m.n-n^2 = 4.m.n$ was obtained algebraically.

Visual proof of the identity $(m + n + k)^2 = m^2+n^2+ k^2 +2. (m. n)+ 2.(m. k)+2.(n. k)$

Step 1

On the AB and AD sides of the ABCD square, whose side length is (m +n + k) unit, m,n and k unit are marked by using a ruler. Then, line segments [GH] and [EF] are drawn respectively on the side [DC] and [AB] parallel to the side [BC] and also, line segments [KL] and [MN] are drawn respectively on the side [BC] and [AD] parallel to the side BC. When drawing as in figure 14, squares with side lengths m, n and k unit are formed on the diagonal [AC] of ABCD square. Rectangles with two S_4, S_5 and S_6 areas each are formed with identical areas above and to the right of the squares on the [AC] diagonal of the ABCD square as seen in figure 14. Let's find firstly the area of square ABCD given in figure 14.

The area of square ABCD was calculated as Area (ABCD) = $(m+n+k).(m+n+k)= (m+n+k)^2$ br^2

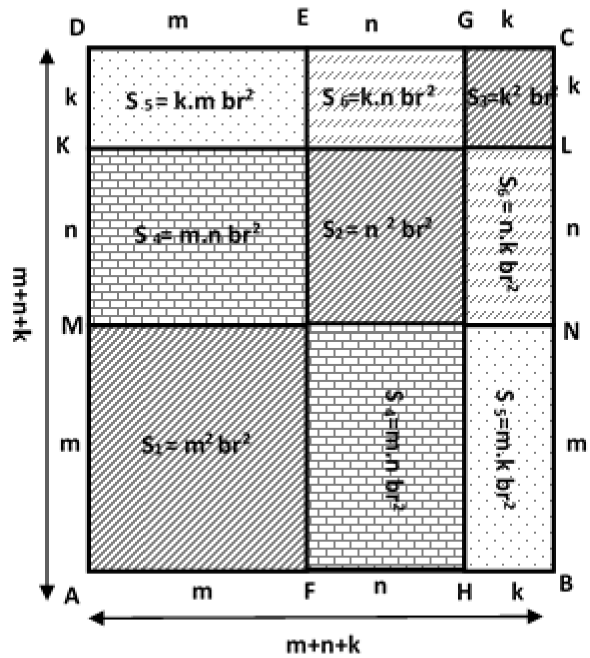

Figure 14. The Visual Proof of $(m+n+k)^2 = m^2+n^2+k^2 +2.(m.n)+ 2.(m.k)+2.(n.k)$

Step 2

Let's show the images of the squares formed on the [AC] diagonal of the ABCD square as in figure 15. The sum of the areas of these squares was found as $T_1= (m^2 + n^2 + k^2) br^2$

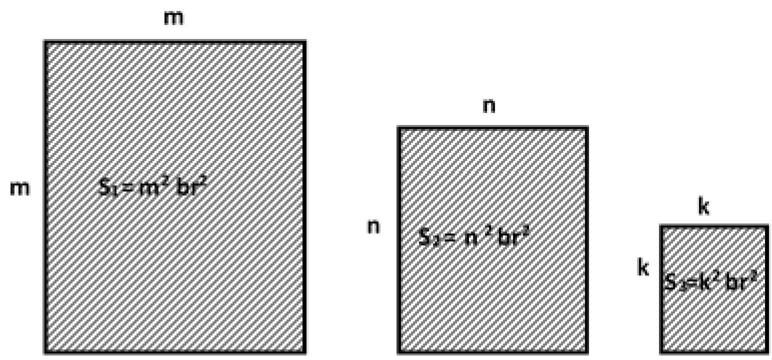

Figure 15. The Visual Representation of the Area of Squares with Sides Respectively m, n and k

Step 3

By using figure 14, the areas of the visual rectangles with side lengths of m and n unit and given in figure 16 are added up. From here, $T_2 = S_4+S_4 =2.S_4=2m.n\ br^2$ was found.

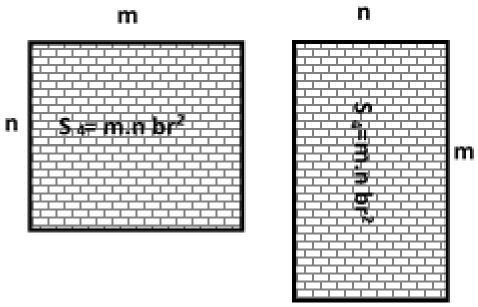

Figure 16. The Visual Representation of the Areas of Identical Rectangle with Sides m and n

Step 4

Using Figure 14, the areas of the visual rectangles given in figure 17 with side lengths m and k unit, and n and k unit are added together.

From here, $T_3 = 2.S_5 + 2.S_6 = 2.(m.k) + 2.(n.k)\ br^2$ was found.

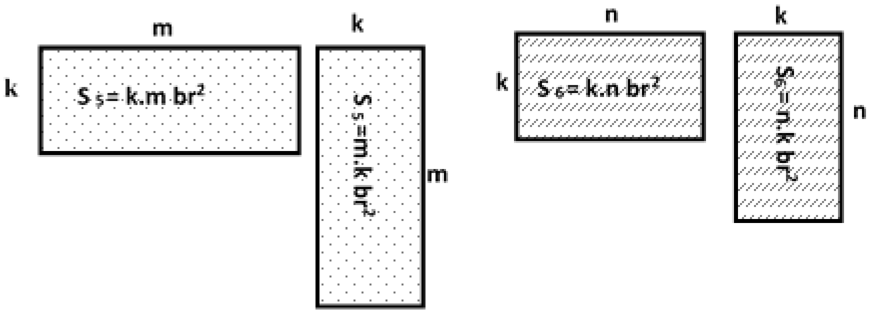

Figure 17. The Visual Representation of the Areas of Identical Rectangle with Sides Respectively

Step 5

As a result; The sum of T_1, T_2 and T_3 areas calculated in figure 15, figure 16 and figure 17 (step 3, step 4 and step 5) were found respectively. When the values found in the algebraic formula Area (ABCD) = $T_1 + T_2 + T_3$ were written by

$$T = T_1 + T_2 + T_3 = (m^2 + n^2 + k^2) + 2.m.n + 2.(k.m) + 2.(k.n)$$

$$= m^2 + n^2 + k^2 + 2.(m.n + m.k + n.k)$$

The sum of these areas found is equal to the area of square ABCD found in step-1 (figure 14).

$$(m + n + k)^2 = (m^2 + n^2 + k^2) + 2.(m.n) + 2.(k.m) + 2.(k.n)$$

$= m^2 + n^2 + k^2 + 2.(m.\ n + k.m + n.\ k)$

So, It was was shown visually

Algebraic proof of the identity $(m + n + k)^2 = (m^2 + n^2 + k^2) + 2.(m.n) + 2.\ (k.m) + 2.(\ k.\ n)$

This $(m + n + k)^2$ algebraic formula could be written as

$(m + n + k).\ (m + n + k).\ (m + n + k) = m.\ (m + n + k) + n.\ (m + n + k) + k.\ (m + n + k)$

If we apply the distributive property of multiplication over addition;

$m.(m + n + k) + n.\ (m + n + k) + k.\ (m + n + k) = m^2 + n^2 + k^2 + 2.\ (m.n) + 2.\ (m.k) + 2.\ (n.k)$ was shown algebraically.

CONCLUSION and DISCUSSION

In this study, visualized algebraic proofs of the formulas of algebraic identities in two dimensions (in the plane) were made. In this direction, visual and algebraic proofs were given in detail and the interested researchers on this subject were informed. The importance of visual proofs and visual representations in mathematics education was understood from related studies. In a study conducted by Demircioğlu and Polat (2015) on pre-service teachers, it was revealed that visual proofs were enjoyable, remarkable, instilling confidence and beneficial to students with learning difficulties.The use of visual representations in the learning and teaching process has a significant impact on the teaching of many courses such as mathematics lessons (Gersten et al., 2009).Students can be encouraged to represent mathematical information in algebraic, oral and written forms, together with visual expression. For students with learning disabilities, teaching concepts or showing proofs of theorems on various visual shapes help students to solve problems more easily, as well as enable students to learn mathematical operations and concepts in a meaningful way (Suh & Moyer, 2007).There are many types of diagrams that can be used effectively to visualize mathematical concepts (Kolloffel et al., 2009) and graphic organizers that enable students with learning disabilities to learn more easily (Dexter & Hughes, 2011). It is thought that the use of visual materials in the lessons may be beneficial. Konyalıoğlu, Aksu and Şenel (2012) also stated in their study that the concept of absolute value is a difficult subject to be understood by students and it would be appropriate for teachers to use a visualization approach in lessons when teaching this subject. The findings obtained from this study show that the use of visualization approach in the lessons makes the lessons easier to learn and the information to be permanent (Tekin, 2010).In their study, Eleftherios and Theodosios (2007) stated that the use of teaching approaches such as visualization and problem-based learning approach is effective in learning mathematical concepts more easily, and at the same time, students have positive attitudes towards mathematics.In their study with secondary school students with learning difficulties, they emphasized that the use of visual representations in lessons provides many advantages and convenience to students when performing mathematical operations and solving problems related to daily life (Yenioğlu et., 2022).The use of the visualization approach at different student levels and learning styles positively supports students' attitudes. Thanks to the visualization approach and problem-based learning, students' positive attitudes towards the lesson are developed (As cited in Abata et al., 2022).In the study, in which the opinions of 10 secondary school students studying at state secondary schools in a town in the Western Black Sea Region were taken about the visuals they used while solving problems, five students stated that they understood and solved the problems more easily by using visualization (Salılmış and Tunç, 2022). In the light of such studies, visual representations and visualization approach have an important place in concretizing abstract mathematical concepts.

Visual proofs enable students to learn abstract concepts meaningfully by creating a link between conceptual knowledge and procedural knowledge. The proofs obtained by visualizing the shapes inform the students about how the formulas are formed, and pave the way for a more permanent learning by avoiding rote methods. In the process of teaching mathematics, it should be pointed out that it is important to give more importance to algebraic and visual proof studies to avoid rote methods within the framework of the causality principle. In the proof processes of algebraic expressions, students may reach erroneous results due to incorrect use of mathematical rules. During the proving process, it may be effective to include visual proofs based on visualization in the lessons concerning the level of the students in eliminating the mistakes and difficulties. The visualization of the concepts and the visual proofs of the theorems by the teacher in the lessons can be a solution to these problems so that the students do not have difficulty in learning the concepts related to algebraic identities.
The opinions of teachers and students should be taken into consideration in terms of the difficulties encountered in the visual proof process and making visual proof in lessons.

Consequently, It is thought that it would be beneficial for mathematics teachers to make proofs by using more visual shapes in the teaching of concepts and algebraic identities in the lessons.

RECOMMENDATIONS

It is suggested that the use of algebra tiles in teaching identities can be useful in lessons, as it will enable concepts to be concretized and visualized.
It is recommended to include more visual models and drawings while proving in mathematics lessons.
In future studies, the effect of visualization approach on students' academic achievement can be investigated.
It can be suggested that mathematics concepts, which are abstract in the lessons, should be associated with daily life as much as possible and explained by using visual models.

ETHICAL TEXT

Ethics Committee Permission Information: Since the study was designed theoretically, ethical permission was not required.

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